CONVAIR (ASTRONAUTICS) DIVISION
GENERAL DYNAMICS CORPORATION

REPORT NO. AC-589
DATE 7 Oct. 1959
NO. OF PAGES 5

CONVAIR | ASTRONAUTICS

CONVAIR DIVISION OF GENERAL DYNAMICS CORPORATION

THE WEAPONS-ASSIGNMENT PROBLEM

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The problem of the title is as follows. There is a stockpile of n missiles and a list of m military targets. The value of the i^{th} target is a known quantity V_i $(1 \leq i \leq m)$. The probability that the i^{th} target will not be hit if attacked by the j^{th} missile is a known quantity P_{ij} . $(1 \leq i \leq m,\ \ 1 \leq j \leq n)$. It is assumed that a hit results in total destruction of the target. What assignment of missiles to targets will effect the maximum expected destruction?

If all missiles were directed to the same target, say the i^{th}, the destruction expected would be

$$D = V_i \left[1 - \prod_{j=1}^{n} P_{ij} \right].$$

This formula is trivial for $n = 1$, and may be proved inductively for other values of n as follows. A strike from a salvo of n missiles can result in three mutually exclusive events:

(a) one of the first (n-1) missiles strikes while the n^{th} misses,

(b) one of the first (n-1) missiles strikes and the n^{th} strikes also,

(c) none of the first (n-1) missiles strikes while the n^{th} strikes,

The probabilities of these three events being added, we obtain, assuming as an induction hypothesis that the formula for D is valid up to n-1:

$$\left(1 - \prod_{j=1}^{n-1} P_{ij}\right) P_{in} + \left(1 - \prod_{j=1}^{n-1} P_{ij}\right) \cdot (1 - P_{in}) + \left(\prod_{j=1}^{n-1} P_{ij}\right) (1 - P_{in}).$$

Algebraically, this is equivalent to $1 - \prod_{j=1}^{n} P_{ij}$, Q.E.D.

Now consider the given data as being laid out tabularly.

V_1	P_{11}	P_{12}	...	P_{1n}
V_2	P_{21}	P_{22}	...	P_{2n}
.	.	.		.
.	.	.		.
.	.	.		.
[illegible]	[illegible]	[illegible]	...	[illegible]

[illegible]

[illegible]

[illegible]

In the next paragraph there is described a relaxation method for the improvement of assignments. This algorithm does not generally yield a solution of the original problem, but is offered here as a means of generating good if not optimal assignments by means of automatic computation. This method, in fact, solves the following simpler problem: as missiles roll off the assembly line, it is desired to assign them optimally and sequentially to targets without altering the assignments of those already stockpiled. There may be instances where the problem just described is the more practical, as when altering the assignment of a missile involves costly reinstrumentation.

Now suppose, for the moment, that an assignment has been fixed for all missiles except one, say the r^{th} one. To which target should the r^{th} missile be directed in order that the resulting full assignment shall be optimal ? This problem can be solved by calculating the m different total destructions that can result when missile r is directed to targets 1, 2,...,m in turn. Having done this for one value of r, we repeat for another, at each step changing one component of the assignment vector to increase the total destruction. This process, repeated indefinitely, is called relaxation.

If all but the r^{th} missile are fired, the probability of survival of target i is

$$\prod_{\substack{j=1 \\ j \neq r}}^{n} H_{ij} .$$

The expected (surviving) value of target i is then

$$V_i \prod_{\substack{j=1 \\ j \neq r}}^{n} H_{ij} .$$

If missile r is fired at target i, an additional expected destruction is effected of magnitude

$$E_i = (1 - P_{ir}) V_i \prod_{j=1}^{n} H_{ij} .$$

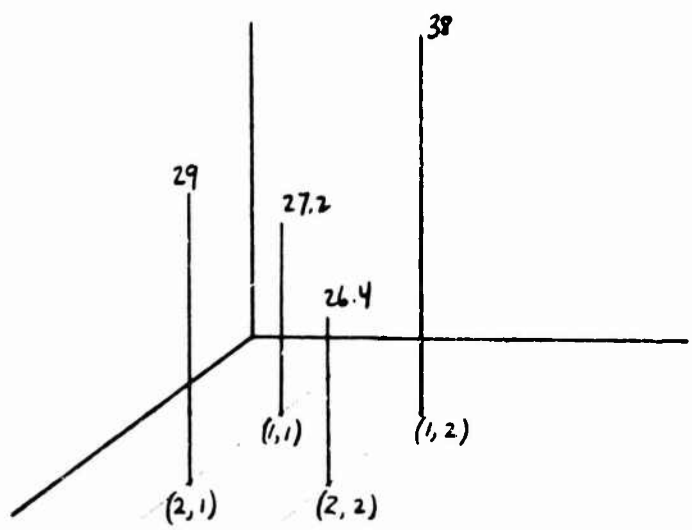
Missile r should then be assigned to that target i for which E_i is a maximum. A program for the IBM 650 which performs this task iteratively is described in Appendix I. We show now by example that the process can fail to produce a solution. Consider the datum

V	P	
40	.5	.8
30	.3	.4

The assignment (2,1) yields a total expected destruction of

$$D = (.7)(30) - (.2)(40) = 29.$$

Any attempt to improve this by alteration of a <u>single</u> component will fail, for D (1,1) = 27.2 and D (2,2) = 26.4. However, D (1,2) = 38. The assignment (1,2) can be obtained from (2,1) only by alteration of <u>two</u> components simultaneously. The example just considered may be represented pictorially as follows. The assignments (1,1), (1,2), (2,1), (2,2) may be shown as points in E_2, and the corresponding value of D may be plotted on the vertical axis.



The relaxation algorithm can proceed from (2,2) to (1,2), or from (1,1) to (1,2), but not from (2,1) to (1,2).

Another example of the same type is as follows:

V	P		
10	.4	.5	.5
20	.8	.7	.5
10	.5	.5	.1

The assignment (2,1,3) yields an expected destruction of

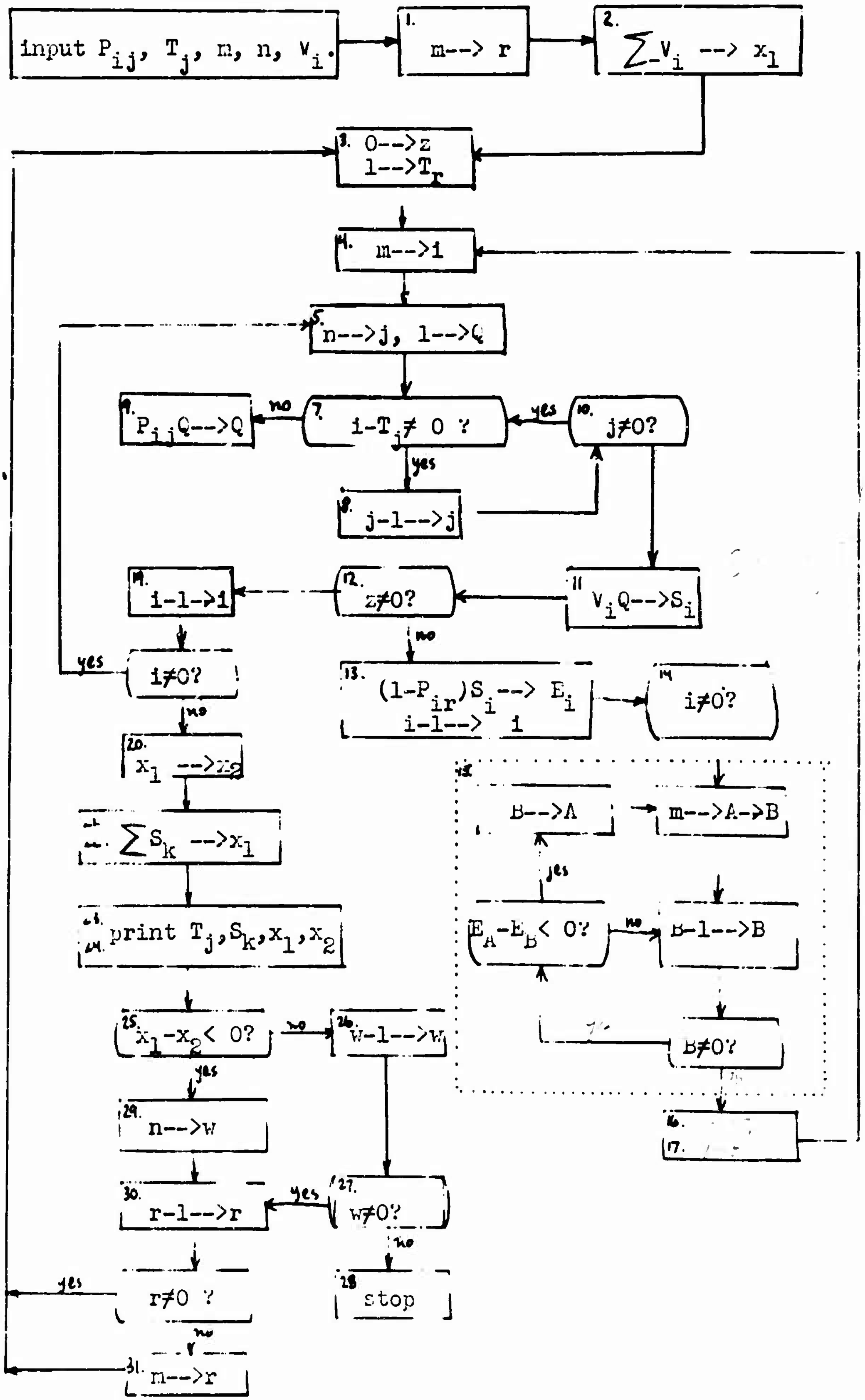
$$D = 20(.2) + 10(.5) + 10(.9) = 18.$$

Every assignment that can be obtained from (2,1,3) by altering a single component yields a lower value of D. However, there exists an assignment, viz. (1,2,3), obtainable from (2,1,3) by altering two components, which yields a higher value of D.

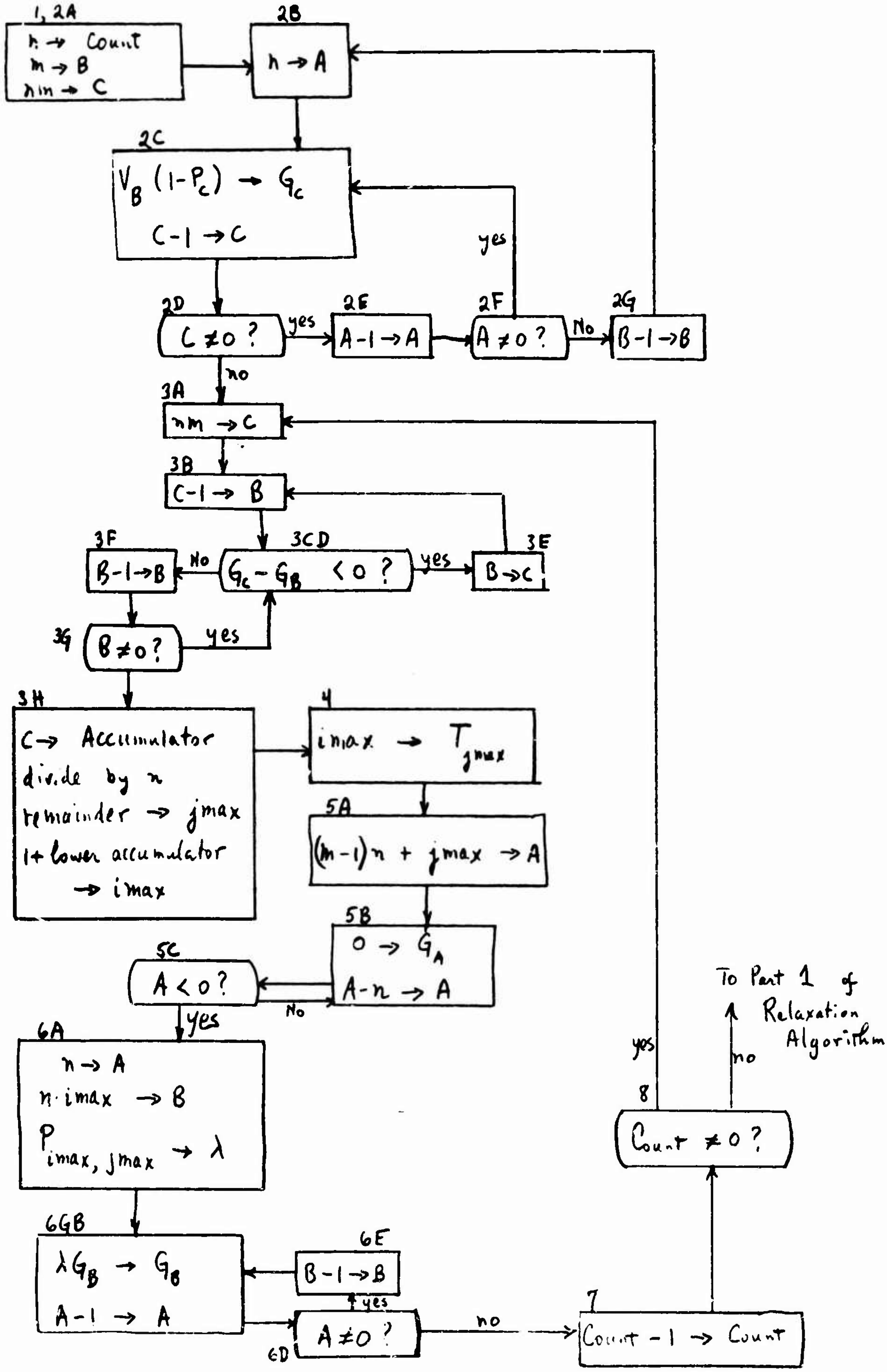
In an attempt to improve an assignment produced by the relaxation method, we may try interchanges of components in the assignment vector T. Both the preceding examples would yield to this treatment. It is not known whether these two methods together will produce optimum assignments in all cases. It seems unlikely that they would. This composite algorithm may be tried over and over on different initial assignments, and these initial assignments may be generated by a random device in the computer. Measures such as this are of use even in simpler problems like the classical assignment problem. In using the relaxation algorithm, it is possible to start with any initial assignment. A reasonable first assignment may be obtained as follows: Calculate all numbers $V_i\ (1 - P_{ij})$. Let the greatest be indexed with $i_o\ j_o$. Then, in the initial assignment, send missile j_o to target i_o. Now reduce V_{i_o} to $V_{i_o}\ P_{i_o j_o}$ and repeat the process, omitting j_o as a value of j. The 650 Program which accomplishes this is described in Appendix II.

All the measures just described must be regarded only as stopgaps until a genuine solution becomes available.

Appendix I: Relaxation Algorithm



Appendix II: Starting Procedure for Relaxation Algorithm



```
                BLR     1950     1999      MISSILE
                REG     P0101    0600      ASSIGNMENT
                REG     V0601    0630      PROBLEM
                REG     T0631    0650
                REG     S0651    0680
                REG     E0681    0710
                REG     Q1027    1036
                REG     X1077    1086
                REG     Y1127    1136
                REG     R1951    1960
                SYN     X1       X0001
                SYN     X2       X0002
                SYN     Z        1027
                SYN     START    1950
                SYN     HALT     0000
                SYN     M        1951
                SYN     N        1952
                REG     G1200    1800
      START     RCD     R0001    1A                     1950   70   1951   0001
      1A        LDD     N                  ALGORITHM    0001   69   1952   0005
                STD     COUNT    2A        TO OBTAIN    0005   24   0008   0011
      2A        LDD     M                  STARTING     0011   69   1951   0004
                RAB      8001              ASSIGNMENT   0004   82   8001   0010
                RAU      8001              FOR          0010   60   8001   0017
                MPY     N                  RELAXATION   0017   19   1952   0022
                RAC      8002    2B        METHOD       0022   88   8002   0031
      2B        LDD     N                               0031   69   1952   0055
                RAA      8001    2C                     0055   80   8001   0061
      2C        RAU     FLONE                           0061   60   0014   0019
                FSB     P0000 C                         0019   33   6100   0027
                FMP     V0000 B                         0027   39   4600   0050
                STU     G0000 C                         0050   21   7199   0002
                SXC      0001    2D                     0002   59   0001   0058
      2D        NZC     2E       3A                     0058   48   0711   0012
      2E        SXA      0001    2F                     0711   51   0001   0067
      2F        NZA     2C       2G                     0067   40   0061   0021
      2G        SXB      0001    2B                     0021   53   0001   0031
      3A        RAU     N                               0012   60   1952   0007
                MPY     M                               0007   19   1951   0072
                RAC      8002    3B                     0072   88   8002   0081
      3B        RAB      6000                           0081   82   6000   0038
                SXB      0001    3C                     0038   53   0001   0044
      3C        RAU     G0000 C                         0044   60   7199   0003
                FSB     G0000 B  3D                     0003   33   5199   0025
      3D        BMI     3E       3F                     0025   46   0028   0029
      3E        LDD      8006                           0028   69   8006   0034
                RAC      8001    3B                     0034   88   8001   0081
      3F        SXB      0001    3G                     0029   53   0001   0035
      3G        NZB     3C       3H                     0035   42   0044   0039
      3H        RAL      8007                           0039   65   8007   0047
                DIV     N                               0047   14   1952   0062
                STU     JMAX                            0062   21   0016   0069
                ALO     ONE                             0069   15   0722   0077
                STL     IMAX     4A                     0077   20   0731   0084
      4A        LDD     JMAX                            0084   69   0016   0719
                RAA      8001                           0719   80   8001   0727
                LDD     IMAX                            0727   69   0731   0734
                STD     T0000 A  5A                     0734   24   2630   0033
      5A        RSU     ONE                             0033   61   0722   0777
                AUP     M                               0777   10   1951   0755
                MPY     N                               0755   19   1952   0772
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                ALO   JMAX                            0772   15  0016  0071
                RAA    8003                           0071   80  8003  0030
                RAU   ZERO     5B                     0030   60  0083  0037
        5B      STU   G0000 A                         0037   21  3199  0052
                LDD   N                               0052   69  1952  0805
                SXA    8001    5C                     0805   51  8001  0761
        5C      BMA   6A       5B                     0761   41  0064  0037
        6A      LDD   N                               0064   69  1952  0855
                RAA    8001                           0855   80  8001  0811
                RAU   N                               0811   60  1952  0057
                MPY   IMAX                            0057   19  0731  0752
                RAB    8002                           0752   82  8002  0861
                SLO   N                               0861   16  1952  0757
                ALO   JMAX                            0757   15  0016  0721
                RAC    8002                           0721   88  8002  0079
                LDD   P0000 C                         0079   69  6100  0053
                STD   LAMBD    6B                     0053   24  0006  0009
        6B      RAU   G0000 B                         0009   60  5199  0753
                FMP   LAMBD                           0753   39  0006  0056
                STU   G0000 B  6C                     0056   21  5199  0802
        6C      SXA    0001    6D                     0802   51  0001  0758
        6D      NZA   6E       7A                     0758   40  0911  0712
        6E      SXB    0001    6B                     0911   53  0001  0009
        7A      RAU   COUNT                           0712   60  0008  0013
                SUP   ONE                             0013   11  0722  0827
                STU   COUNT                           0827   21  0008  0961
                NZU   3A       PT1                    0961   44  0012  0066
        PT1     LDD   N             RELAXATION        0066   69  1952  0905
                STD   R             ALGORITHM         0905   24  0808  1011
                LDD   M             BEGINS            1011   69  1951  0054
                RAA    8001         HERE              0054   80  8001  0060
                SUP    8003    PT2                    0060   11  8003  0717
        PT2     FAD   V0000 A                         0717   32  2600  0877
                SXA    0001                           0877   51  0001  0733
                NZA   PT2                             0733   40  0717  0087
                STU   X1       PT3                    0087   21  1077  0080
        PT3     LDD   ZERO                            0080   69  0083  0036
                STD   Z        PT32                   0036   24  1027  0715
        PT4     LDD   M                               0730   69  1951  0754
                STD   I        PT5                    0754   24  0807  0760
        PT5     LDD   N                               0760   69  1952  0955
                RAA    8001                           0955   80  8001  1061
                RAU   FLONE                           1061   60  0014  0769
                STU   Q        PT7                    0769   21  0024  0927
        PT7     RSU   T0000 A                         0927   61  2630  0085
                AUP   I                               0085   10  0807  1111
                NZU   PT8      PT9                    1111   44  0015  0716
        PT8     SXA    0001    PT10                   0015   51  0001  0771
        PT9     RAU   I                               0716   60  0807  1161
                SUP   ONE                             1161   11  0722  0977
                MPY   N                               0977   19  1952  0822
                ALO    8005                           0822   15  8005  0729
                RAC    8002                           0729   88  8002  0737
                RAU   Q                               0737   60  0024  0779
                FMP   P0000 C                         0779   39  6100  0100
                STU   Q        PT8                    0100   21  0024  0015
        PT10    NZA   PT7      PT11                   0771   40  0927  0075
        PT11    RAU   I                               0075   60  0807  1811
                RAB    8003                           1811   82  8003  0020
                RAU   Q                               0020   60  0024  0829
                FMP   V0000 B                         0829   39  4600  0750
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                STU   S0000 B PT12                  0750  21  4650  0803
       PT12     RAU   Z                             0803  60  1027  0781
                NZU   PT19    PT13                  0781  44  0735  0086
       PT13     RAU   I                             0086  60  0807  1861
                SUP   ONE                           1861  11  0722  1177
                STU   I                             1177  21  0807  0810
                MPY   N                             0810  19  1952  0872
                ALO   R                             0872  15  0808  0063
                RAA    8002                         0063  80  8002  0821
                RSU   P0000 A                       0821  61  2100  1005
                FAD   FLONE                         1005  32  0014  0041
                FMP   S0000 B                       0041  39  4650  0800
                STU   E0000 B PT14                  0800  21  4680  0783
       PT14     RAU   I                             0783  60  0807  1911
                NZU   PT5     PT15                  1911  44  0760  0766
       PT15     LDD   M                             0766  69  1951  0804
                RAA    8001                         0804  80  8001  0860
                RAC    8001   BOX1                  0860  88  8001  0816
       BOX1     SXC    0001                         0816  59  0001  0922
                NZC   BOX2    BOX3                  0922  48  0725  0026
       BOX3     LDD    8005                         0026  69  8005  0032
                STD   K       PT16                  0032  24  0785  0088
       BOX2     RAU   E0000 A                       0725  60  2680  0835
                FSB   E0000 C                       0835  33  6680  0857
                BMI   BOX4    BOX1                  0857  46  0910  0816
       BOX4     RAA    6000   BOX1                  0910  80  6000  0816
       PT16     LDD   R                             0088  69  0808  0762
                RAA    8001                         0762  80  8001  0018
                LDD   K                             0018  69  0785  0738
                STD   T0000 A PT17                  0738  24  2630  0833
       PT17     LDD   ONE                           0833  69  0722  0775
                STD   Z       PT4                   0775  24  1027  0730
       PT19     SXB    0001                         0735  53  0001  0091
                RAU   I                             0091  60  0807  0812
                SUP   ONE                           0812  11  0722  1827
                STU   I                             1827  21  0807  0960
                NZB   PT5     PT20                  0960  42  0760  0714
       PT20     LDD   X1                            0714  69  1077  0780
                STD   X2                            0780  24  1078  0831
                LDD   M                             0831  69  1951  0854
                RAA    8001                         0854  80  8001  1010
                SUP    8003   PT21                  1010  11  8003  0767
       PT21     FAD   S0000 A                       0767  32  2650  1877
                SXA    0001                         1877  51  0001  0883
                NZA   PT21    PT22                  0883  40  0767  0787
       PT22     STU   X1      PT23                  0787  21  1077  0830
       PT23     PCH   X0001   PT24                  0830  71  1077  1927
       PT24     LDD   M                             1927  69  1951  0904
                RAB    8001                         0904  82  8001  1060
                RSC    0000   SET 1                 1060  89  0000  0866
       SET 1    RSA    0008   LOOP1                 0866  81  0008  0972
       LOOP1    LDD   S0001 C                       0972  69  6651  0954
                STD   Y0009 A                       0954  24  3135  0788
                AXC    0001                         0788  58  0001  0094
                SXB    0001                         0094  53  0001  0850
                NZB           THRU1                 0850  42  0853  1004
                AXA    0001                         0853  50  0001  0059
                NZA   LOOP1                         0059  40  0972  0713
                PCH   Y0001                         0713  71  1127  0078
                RSA    0008                         0078  81  0008  0784
                LDD   ZERO    LOOP2                 0784  69  0083  0736
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184   LOOP2   STD   Y0009 A             0736   24   3135   0838
185           AXA    0001               0838   50   0001   0744
186           NZA   LOOP2    SET 1      0744   40   0736   0866
187   THRU1   PCH   Y0001               1004   71   1127   0728
188           RSA    0008               0728   81   0008   0834
189           LDD   ZERO     LOOP3      0834   69   0083   0786
190   LOOP3   STD   Y0009 A             0786   24   3135   0888
191           AXA    0001               0888   50   0001   0794
192           NZA   LOOP3               0794   40   0786   0048
193           LDD   M                   0048   69   1951   1054
194           RAB    8001               1054   82   8001   1110
195           RSC    0000    SET 2      1110   89   0000   0916
196   SET 2   RSA    0008    LOOP4      0916   81   0008   1022
197   LOOP4   LDD   T0001 C             1022   69   6631   0884
198           STD   Y0009 A             0884   24   3135   0938
199           AXC    0001               0938   58   0001   0844
200           SXB    0001               0844   53   0001   0900
201           NZB            THRU2      0900   42   0903   1104
202           AXA    0001               0903   50   0001   0759
203           NZA   LOOP4               0759   40   1022   0763
204           PCH   Y0001               0763   71   1127   0778
205           RSA    0008               0778   81   0008   0934
206           LDD   ZERO     LOOP5      0934   69   0083   0836
207   LOOP5   STD   Y0009 A             0836   24   3135   0988
208           AXA    0001               0988   50   0001   0894
209           NZA   LOOP5    SET 2      0894   40   0836   0916
210   THRU2   PCH   Y0001               1104   71   1127   0828
211           RSA    0008               0828   81   0008   0984
212           LDD   ZERO     LOOP6      0984   69   0083   0886
213   LOOP6   STD   Y0009 A             0886   24   3135   1038
214           AXA    0001               1038   50   0001   0944
215           NZA   LOOP6    PT25       0944   40   0886   0098
216   PT25    RAU   X0001               0098   60   1077   0881
217           FSB   X0002               0881   33   1078   1055
218           BMI   PT29     PT26       1055   46   0858   0809
219   PT26    RAU   W                   0809   60   0862   0817
220           SUP   ONE                 0817   11   0722   0878
221           STU   W                   0878   21   0862   0065
222           STU   Q0001               0065   21   1027   0880
223           PCH   Q0001    PT27       0880   71   1027   0928
224   PT27    NZU   PT30     HALT       0928   44   0931   0000
225   PT29    RAU   M                   0858   60   1951   0813
227           STU   W        PT30       0813   21   0862   0931
228   PT30    RAU   R                   0931   60   0808   0863
229           SUP   ONE                 0863   11   0722   0978
230           STU   R                   0978   21   0808   0912
231           NZU   PT32     PT31       0912   44   0715   0966
232   PT31    LDD   N                   0966   69   1952   1155
233           STD   R        PT3        1155   24   0808   0080
234   PT32    LDD   R                   0715   69   0808   0962
235           RAC    8001               0962   88   8001   0068
236           LDD   ONE                 0068   69   0722   0825
237           STD   T0000 C  PT4        0825   24   6630   0730
238   HALT     01    9999     9999      0000   01   9999   9999
239   ONE      00    0000     0001      0722   00   0000   0001
240   FLONE    10    0000     0051      0014   10   0000   0051
241   ZERO     00    0000     0000      0083   00   0000   0000
```